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NUMERICAL SOLUTION OF COUPLED BURGERS EQUATIONS IN INHOMOGENEOUS FORM

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SUMMARY

A finite difference scheme based on the operator-splitting technique with cubic spline functions is derived for solving the two-dimensional Burgers equations in 'inhomogeneous' form. The scheme is of first-order accuracy in time and second-order accuracy in space direction and is unconditionally stable. The numerical results are obtained with severe/moderate gradients in the initial and boundary conditions and the steady state solutions are plotted for different values of the parameters. It is concluded that the resulting scheme works very well even in the case of very severe gradient in the solution. Also, the general nature of the scheme provides a wider application in the solution of non-linear problems.

KEY WORDS inhomogeneous coupled Burgers equations; operator-splitting technique; cubic spline function; severe/moderate gradients

1. INTRODUCTION

It is known that the Burgers equation is well suited for modelling fluid flows since it incorporates directly the interaction between non-linear convection processes and diffusive viscous processes. In one and two space variables these equations are respectively

$$u_t + uu_x - \frac{1}{Re}u_{xx} = 0, \tag{1}$$

$$u_t + uu_x + vu_y - \frac{1}{Re}(u_{xx} + u_{yy}) = 0, \quad v_t + uv_x + vv_y - \frac{1}{Re}(v_{xx} + v_{yy}) = 0, \tag{2}$$

where Re is the Reynolds number, which often arises in the mathematical modelling of problems in fluid dynamics involving turbulence, and the subscripts denote partial derivatives. The reciprocal of Re is considered to be the kinematic viscosity. These equations possess the desirable attribute that the exact solutions can be constructed readily by invoking the Hopf–Cole transformations.[1] Further, the two-dimensional Burgers equations are an appropriate test case because the equation structure is similar to that of the incompressible fluid flow momentum equations. This system of equations is used in models for the study of hydrodynamical turbulence and wave processes in non-linear media. These equations have been used as a test case for the numerical methods developed for non-lienar problems by several authors (see e.g. References 2–4). Iyengar and Pillai[2] have developed an implicit finite difference scheme based on splines in compression for the one- and two-dimensional Burgers equations in homogeneous form. Arminjon and Beauchamp[3] have solved these equations by the

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method of lines. The splitting-up method with cubic spline functions has been used by Jain and Holla.[4] They have split the first equation of (2) as

$$\frac{1}{4}u_t = -uu_x, \qquad \frac{1}{4}u_t = -vu_y, \qquad \frac{1}{4}u_t = \frac{1}{Re}u_{xx}, \qquad \frac{1}{4}u_t = \frac{1}{Re}u_{yy}$$

and in a similar way have split the second equation of (2). Further, each of the above equations can be approximated in space by cubic spline functions and in time by forward difference operators to obtain a difference scheme. For example, for the first equation they obtained the scheme

$$\left(1 + \frac{\bar{U}^n_{i,j}}{6}\,\delta^2_x(\bar{U}^n_{i,j})^{-1} + \frac{r}{2}\,h\theta_1\bar{U}^n_{i,j}\,\delta_x\right)\left(\bar{U}^{n+1/4}_{i,j} - \bar{U}^n_{i,j}\right) = -\frac{rh}{2}\,\bar{U}^n_{i,j}\,\delta_x\bar{U}^n_{i,j},$$

where $\overline{U}^n_{i,j}$ is the discrete approximation of the velocity component $u(x,y,t)$ at the mesh point (ih,jh,nk), h is the mesh step in directions x and y, k represents the increments in time $(i,j = 0,1,2,\ldots,N; n = 0,1,2,\ldots)$, $\theta_1\epsilon\,[0,1]$ is a cubic spline parameter and

$$\delta_x\bar{U}^n_{i,j} = \bar{U}^n_{i+1,j} - \bar{U}^n_{i-1,j}, \qquad \delta^2_x\bar{U}^n_{i,j} = \bar{U}^n_{i+1,j} - 2\bar{U}^n_{i,j} + \bar{U}^n_{i-1,j}, \qquad r = k/h^2.$$

Similar schemes are derived for the other three equations. Using the approximations

$$(u_x)^n_{i,j} \approx \frac{\mu\delta_x}{2h(1+\delta^2_x/6)}\,\bar{U}^n_{i,j}, \qquad (u_{xx})^n_{i,j} \approx \frac{\delta^2_x}{h^2(1+\delta^2_x/6)}\,\bar{U}^n_{i,j} \tag{3}$$

for space derivatives by involing spline functions, they have solved the aforementioned equations successively, where $\mu\delta_x\overline{U}^n_{i,j} = \overline{U}^n_{i+1,j} - \overline{U}^n_{i-1,j}$. Since both first and second derivatives have $(1 + \frac{1}{6}\delta x^2)$ in the denominator, these terms in equations (3) can be combined together while splitting the equations. In the present investigation we proceed by writing the Burgers equations in matrix form and employ the operator-splitting technique instead of the splitting-up technique to take care of the term $(1+\delta^2_x/6)$.

It is worthwhile remarking that challenging non-linear problems involve high discontinuity and therefore one should choose an appropriate model to take care of non-linearity with initial and boundary conditions having internal or boundary gradients. This will make them more representative of real fluid dynamic problems. Keeping this in view, in this paper we have considered the numerical solution of the coupled Burgers equations in 'inhomogeneous' form in which a non-linear source term is also included. These equations are[1]

$$\begin{aligned} u_t + (u^2)_x + (uv)_y - \frac{1}{Re}\,(u_{xx} + u_{yy}) &= \tfrac{1}{2}Re(u^2+v^2)u,\\ v_t + (vu)_x + (v^2)_y - \frac{1}{Re}\,(v_{xx} + v_{yy}) &= \tfrac{1}{2}Re(u^2+v^2)v. \end{aligned} \tag{4}$$

Even though the exact steady state solution of (4) turns out to be the same as for (2), numerically the problem given by (4) is much more difficult compared with (2) since it incorporates an extra non-linear term. In this paper the numerical results are obtained with severe/moderate gradients in the initial and boundary conditions and the steady state solutions are plotted for different values of the parameters.

The detailed plan of the paper is as follows. In Section 2 the system of inhomogeneous coupled Burgers equations (4) is recast in matrix form. Using a three-step operator-splitting technique with cubic spline functions, the finite difference scheme is derived and it is found that the scheme is unconditionally stable and of first-order accuracy with respect to time and second-order accuracy with respect to space. The numerical results are presented and discussed in Section 3. It is observed that the resulting scheme is efficient and produces satisfactory results in the case of very severe gradients in the solution.

2. DIFFERENTIAL EQUATIONS AND NUMERICAL SCHEME

Recasting the system of inhomogeneous coupled Burgers equations (4) in matrix form, we have

$$U_t + F_x + G_y - \frac{1}{Re}(U_{xx} + U_{yy}) = \frac{1}{2}Re\|U\|^2U, \tag{5}$$

where

$$U = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad F = \begin{bmatrix} u^2 \\ uv \end{bmatrix}, \qquad G = \begin{bmatrix} uv \\ v^2 \end{bmatrix}, \qquad \|U\| = \sqrt{(u^2 + v^2)},$$

subject to the initial conditions

$$U(x,y,0) = U_0(x,y), \quad (x,y)\epsilon D,$$

and the boundary conditions

$$U(x,y,t) = \Psi(x,y,t); \quad x,y\epsilon\partial D, \quad t > 0,$$

where $D = [(x,y,t) : 0 \leq x,y \leq 1, t \geq 0]$ and ∂D is its boundary, $U(x,y,t)$ represents the velocity components $u(x,y,t)$ and $v(x,y,t)$ to be determined and U_0 and Ψ are known functions. Further, the discrete approximation for the velocity components $U(x,y,t)$ at the mesh point ($x_i = ih$, $y_j = jh, t = nk$) is denoted $U_{i,j}^n$ $(i,j = 0,1,2,\ldots,N; n = 0,1,2,\ldots)$, where h is the mesh step in directions x and y and k is the increment in time.

In order to solve equation (5), we use the three-step operator-splitting technique as follows:

$$\tfrac{1}{3}U_t = -F_x + \frac{1}{Re}U_{xx}, \tag{6}$$

$$\tfrac{1}{3}U_t = -G_y + \frac{1}{Re}U_{yy}, \tag{7}$$

$$\tfrac{1}{3}U_t = \tfrac{1}{2}Re\|U\|^2U. \tag{8}$$

Approximating the time derivative by forward differences and the space derivative by the first- and second-order derivatives of the cubic spline function $S_n(x)$ interpolating $F_{i,j}^n$ and $G_{i,j}^n$ $(i,j = 1,2,3,\ldots,k)$, equation (6) becomes

$$\frac{1}{k}(U_{i,j}^{n+1/3} - U_{i,j}^n) = \theta_1 m_{i,j}^{n+1/3} + (1-\theta_1)m_{i,j}^n + \frac{1}{Re}\left[\theta_2 M_{i,j}^{n+1/3} + (1-\theta_2)M_{i,j}^n\right], \tag{9}$$

where $\theta_1, \theta_2 \in [0,1]$ and $m_{i,j}^n$ and $M_{i,j}^n$ denote the first- and second-order derivatives of the cubic spline function $S_n(x)$ respectively.

Now from the condition of continuity of the first and second derivatives of the cubic spline function $S_n(x)$ we have respectively[5]

$$m_{i+1,j}^n + 4m_{i,j}^n + m_{i-1,j}^n = \frac{3}{h}(F_{i+1,j}^n - F_{i-1,j}^n), \tag{10}$$

$$M_{i+1,j}^n + 4M_{ij}^n + M_{i-1,j}^n = \frac{6}{h^2}\,\delta_x^2 U_{i,j}^n. \tag{11}$$

On eliminating the space derivatives $m_{i,j}^n$ and $M_{i,j}^n$ from equations (9)–(11), we obtain the following difference approximation to equation (6):

$$\left(1+\frac{\delta_x^2}{6}\right)\left(U_{i,j}^{n+1/3}-U_{i,j}^{n}\right)=\frac{rh}{2}\left[\theta_1\delta_x F_{i,j}^{n+1/3}+(1-\theta_1)\delta_x F_{i,j}^{n}\right]$$
$$+\frac{r}{Re}\left[\theta_2\delta_x^2 U_{i,j}^{n+1/3}+(1-\theta_2)\delta_x^2 U_{i,j}^{n}\right], \tag{12}$$

where

$$\delta_x U_{i,j}^n = U_{i+1,j}^n - U_{i-1,j}^n, \qquad \delta_x^2 U_{i,j}^n = U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n, \qquad r = k/h^2.$$

In the same way the finite difference approximation to equation (7) is obtained as

$$\left(1+\frac{\delta_y^2}{6}\right)\left(U_{i,j}^{n+2/3}-U_{i,j}^{n+1/3}\right)=\frac{rh}{2}\left[\theta_3\delta_y G_{i,j}^{n+2/3}+(1-\theta_3)\delta_y G_{i,j}^{n+1/3}\right]$$
$$+\frac{r}{Re}\left[\theta_4\delta_y^2 U_{i,j}^{n+2/3}+(1-\theta_4)\delta_y^2 U_{i,j}^{n+1/3}\right], \tag{13}$$

where $\theta_3, \theta_4 \in [0,1]$.

Now for equation (8) we write the finit difference approximation in the form

$$U_{i,j}^{n+1}=\left(1+\frac{k}{2}Re\left\|U_{i,j}^{n+2/3}\right\|^2\right)U_{i,j}^{n+2/3}. \tag{14}$$

The above three equations (12)–(14) are the multistep finite difference formulation of the inhomogeneous coupled Burgers equations given by (5). The intermediate values included in equations (12) and (13) have been taken as

$$U_{i,j}^{n+1/3}=\left(1+\frac{r}{Re}\delta_x^2\right)U_{i,j}^{n}-\frac{rh}{2}\delta_x F_{i,j}^{n}, \quad i=0,\ N, \quad j=0,\ 1,\ 2,\ \ldots,\ N, \tag{15}$$

$$U_{i,j}^{n+2/3}=\left(1+\frac{r}{Re}\delta_y^2\right)U_{i,j}^{n+1/3}-\frac{rh}{2}\delta_y G_{i,j}^{n+1/3}, \quad j=0,\ N, \quad i=0,\ 1,\ 2,\ \ldots,\ N, \tag{16}$$

where δ_x and δ_x^2 are replaced at the lower boundary $i=0$ by $2\Delta_x - \Delta_x^2$ and Δ_x^2 respectively and at the upper boundary $i=N$ by $2\nabla_x + \nabla_x^2$ and ∇_x^2 respectively, where

$$\Delta_x U_{i,j}^n = U_{i+1,j}^n - U_{i,j}^n, \qquad \nabla_x U_{i,j}^n = U_{i,j}^n - U_{i-1,j}^n.$$

Similarly we can write down corresponding expressions for δ_y and δ_y^2.

3. STABILITY AND ACCURACY ANALYSIS OF THE SCHEME

For analysing the stability and accuracy of the scheme, we eliminate the intermediate values, and by performing the necessary simplifications, the scheme finally takes the form

$$W^{n+1} = QW^n + H, \tag{17}$$

where

$$W^n=\begin{bmatrix}\bar{U}_{i,j}^n\\ \bar{V}_{i,j}^n\end{bmatrix}, \qquad Q=\begin{bmatrix}D_1 & 0\\ 0 & D_2\end{bmatrix}.$$

In the amplification matrix Q, D_1 and D_2 are given by

$$D_1 = \frac{\begin{array}{c}\left[1+\left(\frac{1}{6}-\frac{r}{Re}(\theta_2-1)\right)\delta_x^2\right]\left\{1+\bar{U}_{i,j}^n\left[\frac{1}{6}\delta_x^2(\bar{U}_{i,j}^n)^{-1}+rh(\theta_1-1)\mu\delta_x\right]\right\}\\ \times\left[1+\left(\frac{1}{6}-\frac{r}{Re}(\theta_4-1)\delta_y^2\right)\right]\left\{1+\bar{V}_{i,j}^n\left[\frac{1}{6}\delta_y^2(\bar{V}_{i,j}^n)^{-1}+rh(\theta_3-1)\mu\delta_y\right]\right\}\end{array}}{\begin{array}{c}\left[1+\left(\frac{1}{6}-\frac{r\theta_2}{Re}\right)\delta_x^2\right]\left\{1+\bar{U}_{i,j}^n\left[\frac{1}{6}\delta_x^2(\bar{U}_{i,j}^n)^{-1}+rh\theta_1\mu\delta_x\right]\right\}\\ \times\left[1+\left(\frac{1}{6}-\frac{r\theta_4}{Re}\right)\delta_y^2\right]\left\{1+\bar{V}_{i,j}^n\left[\frac{1}{6}\delta_y^2(\bar{V}_{i,j}^n)^{-1}+rh\theta_3\mu\delta_y\right]\right\}\end{array}},$$

$$D_2 = \frac{\begin{array}{c}\left[1+\left(\frac{1}{6}-\frac{r}{Re}(\theta_2-1)\right)\delta_y^2\right]\left\{1+\bar{V}_{i,j}^n\left[\frac{1}{6}\delta_y^2(\bar{V}_{i,j}^n)^{-1}+rh(\theta_1-1)\mu\delta_x\right]\right\}\\ \times\left[1+\left(\frac{1}{6}-\frac{r}{Re}(\theta_4-1)\right)\delta_x^2\right]\left\{1+\bar{U}_{i,j}^n\left[\frac{1}{6}\delta_x^2(\bar{U}_{i,j}^n)^{-1}+rh(\theta_3-1)\mu\delta_x\right]\right\}\end{array}}{\begin{array}{c}\left[1+\left(\frac{1}{6}-\frac{r\theta_2}{Re}\right)\delta_y^2\right]\left\{1+\bar{V}_{i,j}^n\left[\frac{1}{6}\delta_x^2(\bar{V}_{i,j}^n)^{-1}+rh\theta_1\mu\delta_y\right]\right\}\\ \times\left[1+\left(\frac{1}{6}-\frac{r\theta_4}{Re}\right)\delta_x^2\right]\left\{1+\bar{U}_{i,j}^n\left[\frac{1}{6}\delta_x^2(\bar{U}_{i,j}^n)^{-1}+rh\theta_3\mu\delta_x\right]\right\}\end{array}},$$

where $\overline{U}_{i,j}^n$ and $\overline{V}_{i,j}^n$ denote the discrete approximations for the velocity components $u(x,y,t)$ and $v(x,y,t)$ at the mesh point (ih, jh, nk) respectively $(i,j = 0,1,2,\ldots N, n = 0,1,2,\ldots)$, and

$$H = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

in which C_1 and C_2 are given by

$$C_1 = \left(1+\frac{k}{2}Re(A^2+B^2)\right)A, \qquad C_2 = \left(1+\frac{k}{2}Re(A^2+B^2)\right)B,$$

where

$$A = \left(1+\frac{r}{Re}\delta_y^2\right)\left(1+\frac{r}{Re}\delta_x^2\right)\left(1-\frac{rh}{2}\bar{V}_{i,j}^n\delta_y\right)(1-rh\bar{U}_{i,j}^n\delta_x)\bar{U}_{i,j}^n,$$

$$B = \left(1+\frac{r}{Re}\delta_x^2\right)\left(1+\frac{r}{Re}\delta_y^2\right)\left(1-\frac{rh}{2}\bar{U}_{i,j}^n\delta_x\right)(1-rh\bar{V}_{i,j}^n\delta_y)\bar{V}_{i,j}^n.$$

Now, by using the von Neumann criterion of stability,[6] it is found that the diagonal elements of the amplification matrix Q have values less than unity for $\theta_i \geq \frac{1}{2}, i = 1,2,3,4$. Hence the scheme is unconditionally stable for $\theta_i \geq \frac{1}{2}$. It has an accuracy of first order with respect to time and second order with respect to space because of the following expressions of the local truncaction errors in equations (6) and (7) respectively:

$$\left(U_t + \frac{k}{2} U_{xx} + \cdots\right)_{i,j}^{n}$$
$$+ \frac{1}{2}\left[\theta_1\left(F_x + \frac{h}{2} F_{xx} + \frac{h^2}{6} F_{xxx} + \cdots\right)_{i,j}^{n+1/3}\right.$$
$$\left. + (1-\theta_1)\left(F_x + \frac{h}{2} F_{xx} + \frac{h^2}{6} F_{xxx} + \cdots\right)_{i,j}^{n}\right]$$
$$+ \frac{1}{Re}\left[\theta_2\left(U_{xx} + \frac{h^2}{12} U_{4x} + \cdots\right)_{i,j}^{n+1/3}\right.$$
$$\left. + (1-\theta_2)\left(U_{xx} + \frac{h^2}{12} U_{4x} + \cdots\right)_{i,j}^{n}\right], \tag{18}$$

$$\left(U_t + \frac{k}{2} U_{yy} + \cdots\right)_{i,j}^{n+1/3}$$
$$+ \frac{1}{2}\left[\theta_3\left(G_y + \frac{h}{2} G_{yy} + \frac{h^2}{6} G_{yyy} + \cdots\right)_{i,j}^{n+2/3}\right.$$
$$\left. + (1-\theta_3)\left(G_y + \frac{h}{2} G_{yy} + \frac{h^2}{6} G_{yyy} + \cdots\right)_{i,j}^{n+1/3}\right]$$
$$+ \frac{1}{Re}\left[\theta_4\left(U_{yy} + \frac{h^2}{12} U_{4y} + \cdots\right)_{i,j}^{n+2/3}\right.$$
$$\left. + (1-\theta_4)\left(u_{yy} + \frac{h^2}{12} U_{4y} + \cdots\right)_{i,j}^{n+1/3}\right]. \tag{19}$$

4. NUMERICAL RESULTS AND DISCUSSION

With the help of the aforementioned difference scheme we have solved the coupled Burgers equations in inhomogeneous form (4). For initial and boundary conditions as mentioned earlier we observed that the exact solution of the two-dimensional Burgers equations can be generated by making use of the Hopf–Cole transformations[1]

$$u = -\frac{2}{Re}\frac{\phi_x}{\phi}, \qquad v = -\frac{2}{Re}\frac{\phi_y}{\phi}, \tag{20}$$

where ϕ is the solution of

$$\phi_t = \phi_{xx} + \phi_{yy}.$$

In the present study we have determined only steady state solutions of the coupled Burgers equations under consideration. From (20) we get the general expressions for u and v as

$$u = -\frac{2}{Re}\frac{a_1 + a_3y + \alpha a_4\{\exp[\alpha(x-x_0)] - \exp[-\alpha(x-x_0)]\}\cos(\alpha y)}{a_0 + a_1x + a_2y + a_3xy + a_4\exp[\alpha(x-x_0)] + \exp[-\alpha(x-x_0)]\sin(\alpha y)}, \tag{21}$$

$$v = -\frac{2}{Re}\frac{a_2 + a_3x - \alpha a_4\{\exp[\alpha(x-x_0)] - \exp[-\alpha(x-x_0)]\}\sin(\alpha y)}{a_0 + a_1x + a_2y + a_3xy + a_4\exp[\alpha(x-x_0)] + \exp[-\alpha(x-x_0)]\cos(\alpha y)}, \tag{22}$$

where $a_0, a_1, a_2, a_3, a_4, \alpha$ and x_0 can be chosen to give specific features to the flow. The exact steady state solutions of the coupled Burgers equations in inhomogeneous form are given by (21) and (22) and this inhomogeneous form is solved numerically with Dirichlet boundary conditions given by (21) and (22) for appropriate choices of the parameters $a_0, a_1, a_2, a_3, a_4, \alpha$ and x_0 and plotted.

Figures 1 and 2 show plots of u and v respectively for the parameter values

$$a_0 = a_1 = 110{\cdot}13, \quad a_2 = a_3 = 0, \quad a_4 = 1{\cdot}0, \quad \alpha = 5, \quad x_0 = 1, \quad Re = 10,$$

with mesh sizes h = 1/25 and r = 1·5. This is the steady state solution of the coupled Burgers equations in inhomogeneous form which is achieved at t=0·002560. It is clear from Figures 1 and 2 that the solution has a sharp jump on one side for u as well as v but is otherwise smooth. Figures 3 and 4 depict the steady state solution for the parameter values

$$a_0 = a_1 = 0{\cdot}011013, \quad a_2 = a_3 = 0, \quad a_4 = 1{\cdot}0, \quad \alpha = 5, \quad x_0 = 1, \quad Re = 10.$$

In this case we have found severe gradients at initial points.

Upon comparing this scheme with the scheme due to Jain and Holla,[4] we find that their scheme does not yield satisfactory results in the case of the gradients considered above. They have claimed that their results give a good approximation up to intermediate Reynolds numbers when using continuous initial conditions. However, when we apply the same scheme to the above-mentioned cases, it does not give

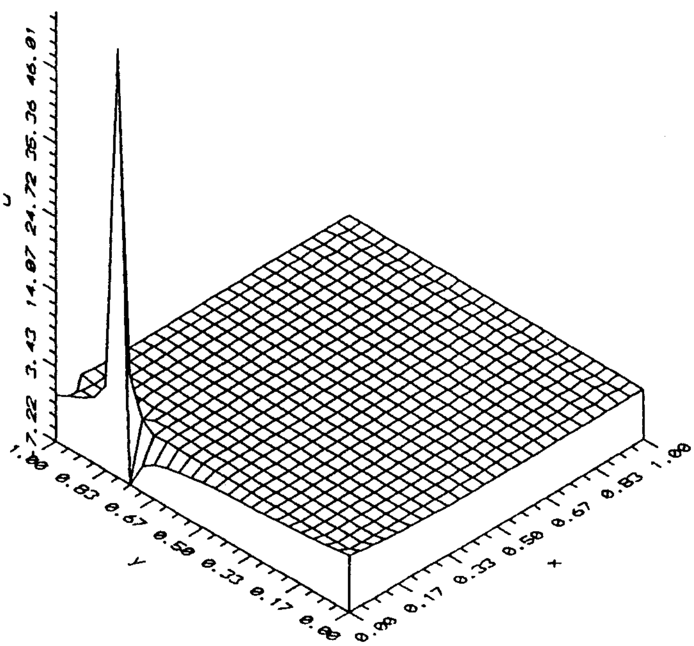

Figure 1. Plot of u at steady state for first set of parameter values

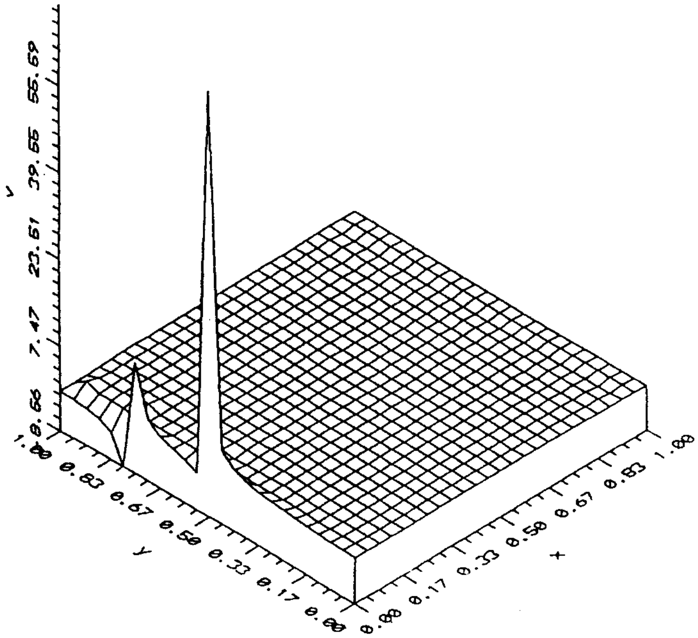

Figure 2. Plot of v at steady state for first set of parameter values

any convergent solution for the given parameters, but when we reduce the sharpness in the conditions, their scheme works. Hence we can say that our scheme yields good results even in the presence of severe gradients in the initial/boundary conditions.

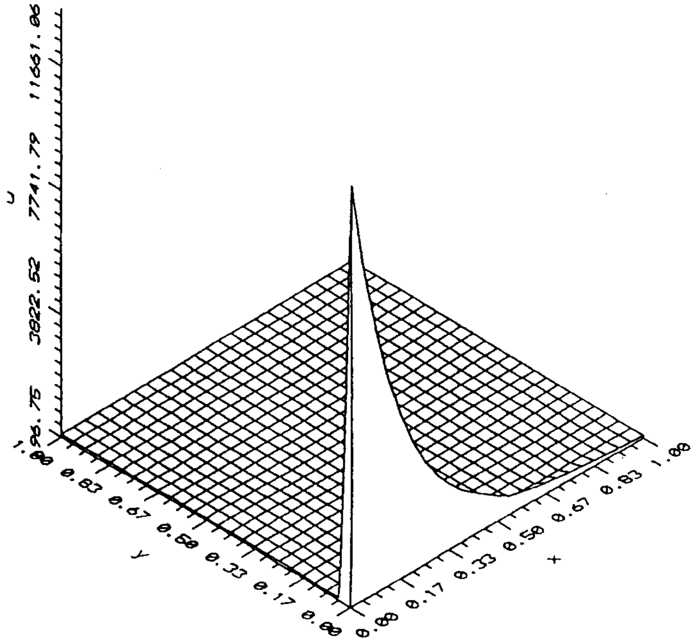

Figure 3. Plot of u at steady state for second set of parameter values

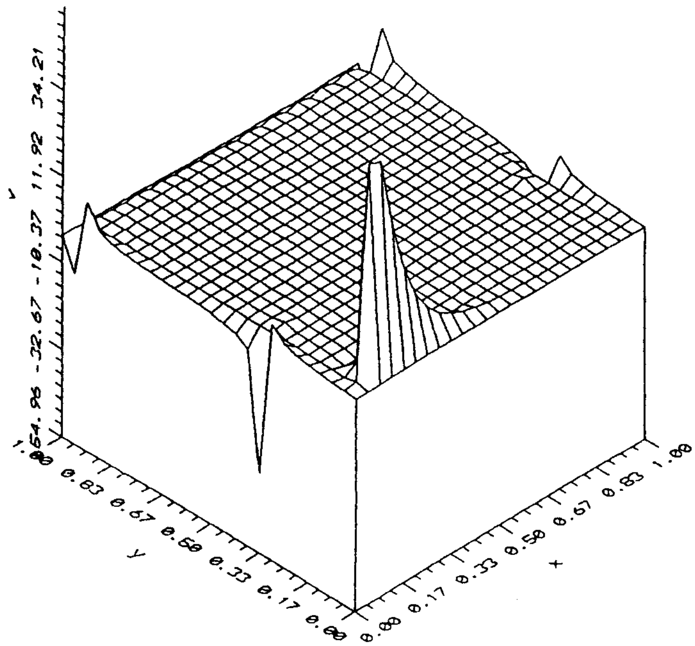

Figure 4. Plot of v at steady state for second set of parameter values

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APPENDIX: NOMENCLATURE

t	time
u, v	velocity components in directions x and y
x, y	Cartesian co-ordinates
$\|\cdot\|$	norm
Greek letters	
δ	central difference operator
μ	averaging operator
Δ	forward difference operator
∇	backward difference operator
θ_i	cubic spline parameter (i = 1,2,3,4)

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